R- ANNIHILATOR -HOLLOW AND R- ANNIHILATOR LIFTING MODULES

Sahira M. Yaseen
Sahira.mahmood@gmail.com
Mathematics Department, College of Science, University of Baghdad, Iraq

ABSTRACT: *Let M be a unitary left R- module on associative ring with identity R . A submodule K of M is called R- ann-small if K+T=M ,where T is a submodule of M , implies that ann(T)=0, where ann(T) indicates annihilator of T in R . In this paper we introduce the concepts R- annihilator -hollow modules, R- annihilator -lifting modules and R- annihilator –amply supplemented modules . We give many properties related with this type of modules.*

Keywords: hollow module, lifting module, Small submodule, R- ann-hollow and R- ann-lifting modules

INTRODUCTION

Let M be a unitary left R- module on associative ring with identity R . M is called a hollow module if every proper submodule is small in M ,where a submodule N of R-module M is called small in M ($N \ll M$) if $N + K \neq M$ for each proper submodule K of M. A proper submodule N of R-module M is called an essential in M ($N \leq^e M$) if for every

nonzero submodule K of M then $N \cap K \neq 0$ [1].

In [2] there was given the concept of R- ann- small submodules . A submodul N of a R- module M is called R- ann- small if N+T=M, T a submodule of M , implies that $ann_R(T) = 0$, where $ann_R(T) = \{r\ R: r.T=0\}$ .In this paper we give the concept of R- annihilator -hollow and R-annihilator -lifting modules , Let U and V be submodules of an R-module M. We say that M is R-annihilator-hollow module if every proper submodule of M is R-a-small in M. And R-module M is called R- annihilator –lifting if for any submodule N of M there exist submodules K , K' of M such that $M = K \oplus K'$ with $K \leq N$ and $N \cap K'$ is R-annihilator -small in N and we introduce the concept R-annihilator –amply supplemented .Some properties of these modules are considered and give some characterizations for such modules.

1. R- annihilator-hollow module

Now we recall the definition and properties of R- ann -small submodule and introduce the definition of R-annihilator -hollow module .

Definition 1.1:[2] A submodule N of a module M is R-annihilator - small in M(R-a-small) if N+X=M , X a submodule of M ,implies that annX=0 , we write $N \ll^a M$ in this case .

Remarks1.2:[2]

1- Let A and B be submodules of M such that $A \leq B$. if $A \ll^a B$, then $A \ll^a M$.

2- Let A and B be submodules of a module M such that $A \leq B$. If $B \ll^a M$, Then $A \ll^a M$

3- Let $f: M \rightarrow N$ be an epimorphism . If $H \ll^a N$, then $f^{-1}(H) \ll^a M$.

4- Let $M = M_1 \oplus M_2$ If N_1 and N_2 are R-a-small submodules of M_1, M_2 respectively thus $N_1 \oplus N_2$ is R-a-small submodule of $M_1 \oplus M_2$.

Now we can prove the following

Proposition1-3:- Let $M = D_1 \oplus D_2$ such that ann $D_2 \leq^e R$, and $A \leq D_1$. If $A \ll^a M$, Then $A \ll^a D_1$.

Proof:- Let $M = D_1 \oplus D_2$, and $A \leq D_1$, to show that $A \ll^a D_1$ set $M_1 = A+B$, then $M = A+B+D_2$,$A \ll^a M$ then $ann(B+D_2)=0$, $ann(B+D_2) = ann(B) \cap ann(D_2)$ but ann $D_2 \leq^e R$ thus $ann(B)=0$ and $A \ll^a D_1$.

Corollary1-4:- Let $M = D_1 \oplus D_2$ such that ann $D_1 \leq^e R$ and ann $D_2 \leq^e R$, and let $N \leq M$,$N = N_1 \oplus N_2$. If $N \ll^a M$ then $N_1 \ll^a D_1$ and $N_2 \ll^a D_2$.

Definition 1.5 :A nontrivial module M is called R-annihilator -hollow(R- a-hollow) if every proper submodule of M is R-a-small in M.

Examples and Remarks 1.6 :

(1) Z as Z- module is R- a-hollow module but it is not hollow.

(2) Z_6 and Z_4 as Z- module are not R- a- hollow modules.

(3) Simple modules are hollow module but not R- a-hollow modules.

(4) If M a torsion free module on integeral domain R,then M is R- a- hollow.

(5)- If M be a faithful R-module , then R- a-hollow and hollow are equivalent.

Proposition 1-7:- Let M be a faithful module on R, and ann(N) is essential for every N be a submodule of M ,then M is R- a-hollow module.

Proof:-Let D be submodule of M ,then by (prop.2. 1.13[2]) ,D is R-a-small submodule.

The epimorphic image of ;R-a-hollow module need not be R- a- hollow as the following example shows:-

Consider Z and Z_4as Z – modules and $\pi: Z \rightarrow Z_4$ let be the natural epimorphism . Z as Z- module is R- a-hollow module ,$\{0\} \ll^a Z$. But $\pi(\{0\})=0$ is not Z-a-small in Z_4 , where

$Z_4 = 0 + Z_4$ and ann $Z_4 = 4Z \neq 0$.

Proposition1-8:-Let M and N be two modules on R , and $f: M \rightarrow N$ be an epimorphism .If N is R-a-hollow module then M is R-a-hollow module.

Proof:-Let K be submodule of M ,then f(K) is submodule of N and since N is R-a-hollow module , f(K) is R-a-small submodule, then $f^{-1}(f(k)) \ll^a M$ by (Remark1.2(3)), $f^{-1}(f(k)) = K + kerf$, $K \leq K + kerf$ then by(Remark 1.2(1)), K is R-a-small submodule of M.

Corollary 1.9: Let M be a module on R , D be submodule of M . If M/D is R- a-hollow module then M is R- a-hollow module.

A submodule N of R-module M is called fully invariant submodule of M if $f(N) \subseteq N$, for every $f \in Hom(M,M)$ .A module M is called duo module if every submodule of M is fully invariant[3].

Proposition 1.10: Let $M = D_1 \oplus D_2$ be duo module .If D_1 and D_2 are R- a-hollow modules, then M is R- a-hollow module.

Proof: Let $D_1$and D_2 be R- a-hollow modules,and $N_1 \oplus N_2$be a proper Submodule of $D_1 \oplus D_2$

$N_1 \leq D_1$and $N_2 \leq D_2$, then $N_1$and N_2 are R-a-small submodules of D_1, D_2 respectivly thus by(Remark 1-2(4)) $N_1 \oplus N_2$ is R-a-small submodule of M.

Corollary1-11: Let $M= D_1 \oplus D_2$ be an R- module such that $R= ann(D_1)+ann(D_2)$.If D_1 and D_2 are be R- a-hollow , then so is M.

ring R is called R- a-hollow, if R is R- a-hollow R-module.

A module M is multiplication , if for every submodule F of M there exists an ideal I of R such that F = IM = (F:M)M [4].

Proposition 1.12 : Let M be a multiplication R-module. If M is R- a-hollow then R is a R- a-hollow ring.

Proof: Suppose that M is R- a-hollow. Let I be an ideal of R. Then IM is a submodule of M and hence IM is R-a-small ([2], Prop.2.1.17). then I is R-a-small ideal of R and hence R is R- a-hollow.

2. R- annihilator-lifting module :

M is lifting module if for any submodule N of M there exist submodules L, K of M such that $M=L\oplus K$ with $L\leq N$ and $N\cap K \ll N$ (equivalently $N\cap K\ll M$)[5]. In this section we introduce the notion of R- annihilator-lifting module(R- a-lifting) and discus some properties of this kind of modules.

Definition 2.1 : M is called R- annihilator-lifting (R- a-lifting) if for any submodule N of M there exist submodules L , K of M such that $M=L\oplus K$ with $L\leq N$ and $N\cap K \ll^a N$.

By using Remark (1.2) we get the following remark.

Remark 2.2 : A module M is R- a-lifting if and only if for any submodule N of M there exist submodule L of M such that $M=L\oplus K$ and $N\cap K \ll^a M$.

We shall call a ring R , R- a-lifting if R is a R- a-lifting as an R- module. The following Proposition gives a characterization of R- a-lifting modules.

Proposition 2-3: Let M be a R- a-lifting then every submodule N of M, can be written as $N=A\oplus B$ where A is direct summand of M and $B\ll^a M$.

Proof: -is trivial**.**

Remark 2.4 : Every R- a-hollow module is R- a-lifting module.

Proof: - let N be submodule of M if $N\neq M$, R- a-hollow module then $N\ll^a M$,

$M= (0) \oplus M$,with $(0)\leq N$ and $N\cap M \ll^a M$.

Proposition 2.5 : Let $M=H_1\oplus H_2$ be duo module .If H_1 and H_2 are R- a-lifting modules, then M is R- a -lifting module.

Proof:

let H_1 and H_2 are R-a-lifting modules, let N submodule of M ,then $N= (N\cap H_1)\oplus(N\cap H_2)$ .For each $i\in \{1,2\}$,there exists a direct summund D_i of H_i ,such that $H_i= D_i \oplus L_i$ with $D_i \leq N\cap Hi$ and $N\cap L_i \ll^a L_i$ then , $M= (D_1\oplus L_1)\oplus (D_2\oplus L_2)= (D_1\oplus D_2)\oplus (L_1\oplus L_2)$,we have$(D_1\oplus D_2) \leq N$,and $N\cap(L_1\oplus L_2)\ll^a (L_1\oplus L_2)$by(Remark 1-2) then M is R- a-lifting module.

Corollary2-6: Let $M=H_1\oplus H_2$ be a module such that $R=ann(H_1)+ann(H_2)$.If H_1 and H_2 are R- a-lifting modules, then so is M.

Theorem 2.7 : Let M be a multiplication R -module. if M is R- a-lifting module ,then R is R- a-lifting ring .

Proof: Assume that M is R- a-lifting module . Let I be an ideal of R. Then N = IM is a submodule of M, hence there exist submodules K and K' of M with $K \subseteq N$, $M = K \oplus K'$ and

$(N \cap K'))\ll^a M$ (bydef.2.1). But M is a multiplication R module, so there are ideals J and J' of R such that K = JM and K' = J'M. We get $J \subseteq I$ (since $K \subseteq N$). We have $M = K\oplus K' = JM\oplus J'M = (J\oplus J')M$ implies that $R = J\oplus J'$.Now, $N \cap K' = (IM \cap J'M) \ll^a M$ and since $(J\cap J')M \subseteq IM \cap J'M$ it follows that $(J\cap J')M \ll^a M$ (Remark 1-2) and according to .([2]cor.2.1.18), we get $[(J\cap J')M:M] \ll^a R$. But$[(J\cap J')M:M] = I \cap J'$, therefore $(I\cap J')\ll^a R$ then R R- a-lifting ring .

3-R- ann-amply supplemented module:-

In [6] Al-Hurmuzy and Al-Bahrany**,** introduce the concept of R- annihilator supplemented (R-a-supplemented) module . In this section we introduce the concept of R- annihilator amply supplemented (R-a-amply supplemented) module, We also give some basic properties of this class of modules.

Definition 3. 1[6]: Let V and U be submodules of an R-module M. V is R-annihilator-supplement (R-a-supplement;) of U in M if M=U+V and whenever $Y\leq V$ and M=U+Y, then ann Y=0.

Let M be R-module. M is R-annihilator -supplemented(R-a-supplemented) module if every proper submodule of M has R-a-supplement .

Proposition 3.2[6]: Let U and V be submodules of R-module M. Then V is R-a-supplement of U if and only if M=U+V and $U\cap V \ll^a V$.

Now we introduce the following concept

Definition 3. 3:- Let M be an R-module. M is R-annihilator –amply supplemented(R-a-amply supplemented) module if for any submodules A,B of M with M=A+B there exists an R-a-supplement K of A such that $K \leq B$.

Proposition 3.4: Let M and N be R-modules and let $f: M\rightarrow N$ be an epimorphism if N is R-a-amply supplemented module, then M is R-a-amply supplemented module.

Proof: Let A, B be submodules of M with M=A+B, then N=f(A)+f(B)since N is R-a-amply supplemented module, there exists a submodule K of N such that N= f(A)+K , $f(A) \cap K \ll^a K \leq f(B)$. $M= f^{-1}(N)=)$, $f^{-1}(f(A)+K)=A+ker f + f^{-1}(K))=A + f^{-1}(K)$. Since $f(A) \cap K \ll^a K$ then by (Remark1-2) , $f^{-1}(f(A) \cap K) \ll^a f^{-1}(K)$ but $f^{-1}(f(A) \cap K)=(A+ ker f)\cap f^{-1}(K)= ker f +(A\cap f^{-1}(K)) \ll^a f^{-1}(K)$ by (Remark1-2) $A\cap f^{-1}(K) \ll^a f^{-1}(K)$ so $f^{-1}(K)$ is R-a- supplement of A and $K \leq f(B)$ then $f^{-1}(K) \leq B$ thus M is R-a-amply supplemented .

Proposition 3.5: Let M be an R-modules .If every submodule of M is an R-a- supplemented module, then M is an R-a-amply supplmented module.

Proof: Let A, B be submodules of M with M=A+B, by assumption there is $H\leq B$ such that ,$B=(A \cap B)+H$ and $(A \cap B) \cap H =A \cap H \ll^a H$. Thus $B=(A \cap B)+H \leq H+A$, since $M=A+B\leq H+A$ then M=A+H.

Corollary 3.6: For any ring R, the following are equivalent

1-ALL modules are R-a-amply supplemented module.

2-ALL modules on R are R-a- supplmented module.

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